

# SAMPLING MEASURES, MUCKENHOUTP HAMILTONIANS, AND TRIANGULAR FACTORIZATION

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ABSTRACT. Let  $\mu$  be an even measure on the real line  $\mathbb{R}$  such that

$$c_1 \int_{\mathbb{R}} |f|^2 dx \leq \int_{\mathbb{R}} |f|^2 d\mu \leq c_2 \int_{\mathbb{R}} |f|^2 dx$$

for all functions  $f$  in the Paley-Wiener space  $PW_a$ . We prove that  $\mu$  is the spectral measure for the unique Hamiltonian  $\mathcal{H} = \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix}$  on  $[0, a]$  generated by a weight  $w$  from the Muckenhoupt class  $A_2[0, a]$ . As a consequence of this result, we construct Krein's orthogonal entire functions with respect to  $\mu$  and prove that every positive, bounded, invertible Wiener-Hopf operator on  $[0, a]$  with real symbol admits triangular factorization.

## 1. INTRODUCTION

The classical Paley-Wiener space  $PW_a$  consists of entire functions of exponential type at most  $a$  square summable on the real line,  $\mathbb{R}$ . A measure  $\mu$  on  $\mathbb{R}$  is called a sampling measure for the space  $PW_a$  if there exist positive constants  $c_1, c_2$  such that

$$c_1 \int_{\mathbb{R}} |f|^2 dx \leq \int_{\mathbb{R}} |f|^2 d\mu \leq c_2 \int_{\mathbb{R}} |f|^2 dx, \quad f \in PW_a. \quad (1)$$

Let  $\mathcal{H}$  be a regular Hamiltonian on  $[0, a]$ , that is,  $\mathcal{H}$  is a mapping from  $[0, a]$  to the set of  $2 \times 2$  non-negative matrices with real entries such that  $\text{trace } \mathcal{H}$  is a positive non-vanishing function in  $L^1[0, a]$ . Denote by  $\Theta_{\mathcal{H}} = \Theta_{\mathcal{H}}(r, z)$  solution of the following Cauchy problem:

$$JX'(r) = z\mathcal{H}(r)X(r), \quad X: [0, a] \rightarrow \mathbb{C}^2, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z \in \mathbb{C}. \quad (2)$$

It is known from a general theory of canonical Hamiltonian systems that for every measure  $\mu$  satisfying (1) there exists a regular Hamiltonian  $\mathcal{H}$  with  $\int_0^a \sqrt{\det \mathcal{H}} = a$  such that  $\mu$  is a spectral measure for problem (2). The latter means that the Weyl-Titchmarsh transform

$$\mathcal{W}_{\mathcal{H}, a}: X \mapsto \frac{1}{\sqrt{\pi}} \int_0^a \langle \mathcal{H}(r)X(r), \Theta_{\mathcal{H}}(r, \bar{z}) \rangle_{\mathbb{C}^2} dr, \quad z \in \mathbb{C}, \quad (3)$$

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generated by solution  $\Theta_{\mathcal{H}}$  of Cauchy problem (2) maps isometrically the space

$$L^2(\mathcal{H}, a) = \left\{ X : [0, a] \rightarrow \mathbb{C}^2 : \|X\|_{L^2(\mathcal{H}, a)}^2 = \int_0^a \langle \mathcal{H}(r)X(r), X(r) \rangle_{\mathbb{C}^2} dr < \infty \right\} / \mathcal{K}(\mathcal{H}),$$

$$\mathcal{K}(\mathcal{H}) = \left\{ X : \mathcal{H}(t)X(t) = 0 \text{ for almost all } t \in [0, r] \right\}$$

into the space  $L^2(\mu)$ . A general problem in the inverse spectral theory is to translate properties of a spectral measure  $\mu$  into properties of the Hamiltonian  $\mathcal{H}$  it generates.

Two essentially different cases of the above problem attracted much attention. If  $\mu$  is a “small perturbation” of the Lebesgue measure on  $\mathbb{R}$  (in the sense that the Fourier transform of  $\mu$  restricted to the interval  $[-a, a]$  differs from the point mass measure  $\delta_0$  concentrated at 0 by a function in  $L^1[-a, a]$ ), the I. M. Gelfand–B. M. Levitan approach [6], [12] gives a quite precise information on relation between  $\mu$  and  $\mathcal{H}$ . On the other hand, if  $\mu$  is arbitrary measure on  $\mathbb{R}$  such that  $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$ , the theory of M. G. Krein [8] (for even measures  $\mu$ ) and L. de Branges [4] (for all  $\mu$ ) implies the *existence* of a unique Hamiltonian  $\mathcal{H} \in L^1_{\text{loc}}[0, \infty)$  such that  $\mu$  is the spectral measure for  $\mathcal{H}$ . However, it is not known how translate even simple properties of a Hamiltonian  $\mathcal{H}$  (e.g., membership in  $L^p$  class for some  $p > 1$ ) to the properties of its spectral measure  $\mu$  and vice versa. In this paper we consider a “median” situation (spectral measures with sampling property (1) for the Paley-Wiener space  $\text{PW}_a$ ) and use both Gelfand-Levitan and Krein-de Branges theories.

A measure  $\mu$  on  $\mathbb{R}$  is called even if  $\mu(S) = \mu(-S)$  for every Borel set  $S \subset \mathbb{R}$ . A function  $w > 0$  belongs to the Muckenhoupt class  $A_2[0, a]$  if the supremum of products  $(\frac{1}{|I|} \int_I w) \cdot (\frac{1}{|I|} \int_I \frac{1}{w})$  over all intervals  $I \subset [0, a]$  is finite. Here is the main result of the paper.

**Theorem 1.** *Let  $\mu$  be an even sampling measure for  $\text{PW}_a$ . Then  $\mu$  is the spectral measure for problem (2) corresponding to the unique Hamiltonian  $\mathcal{H} = \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix}$  generated by a weight  $w \in A_2[0, a]$ .*

The Hamiltonian  $\mathcal{H}$  in Theorem 1 could be recovered from the spectral measure  $\mu$  by means of the following simple formula:

$$w(r) = \pi \frac{\partial}{\partial r} \left\| T_{\mu, r}^{-1} \text{sinc}_r \right\|_{L^2(\mu)}^2, \quad \text{sinc}_r = \frac{\sin rx}{\pi x}, \quad r \in [0, a],$$

where  $T_{\mu, r}$  is the truncated Toeplitz operator on  $\text{PW}_r$  with symbol  $\mu$  defined by

$$(T_{\mu, r} f)(z) = \int_{\mathbb{R}} f(x) \frac{\sin r(x-z)}{\pi(x-z)} d\mu(x), \quad z \in \mathbb{C}. \quad (4)$$

A nontrivial fact is that the continuous increasing function  $r \mapsto \|T_{\mu, r}^{-1} \text{sinc}_r\|_{L^2(\mu)}^2$  is *absolutely* continuous and its derivative  $w/\pi$  does not vanish on a set of positive Lebesgue measure. In the proof of Theorem 1 we first obtain an estimate for the “ $A_2$ -norm” of  $w$  in terms of  $c_1, c_2$  assuming above properties of  $w$ ; then use an approximation argument based on a description of positive truncated Toeplitz operators on  $\text{PW}_r$  and  $L^p$ -summability of weights  $w \in A_2[0, a]$  for some  $p > 1$ .

Section 5 in [2] contains an example of a diagonal Hamiltonian  $\mathcal{H}$  on  $[0, 1]$  such that both  $\mathcal{H}, \mathcal{H}^{-1}$  are uniformly bounded on  $[0, 1]$ , but the spectral measures of the corresponding problem (2) fail to have sampling property. This shows that  $A_2[0, a]$

class does not describe canonical Hamiltonian systems generated by sampling measures for  $\text{PW}_a$ .

Theorem 1 yields two results of independent interest.

Given a measure  $\mu$  satisfying (1) and a number  $r \in [0, 2a]$ , denote by  $(\text{PW}_{[0,r]}, \mu)$  the Paley-Wiener space of functions from  $L^2(\mathbb{R})$  with Fourier spectrum in  $[0, r]$  equipped with the inner product taken from  $L^2(\mu)$ .

**Theorem 2.** *Let  $\mu$  be an even sampling measure for the space  $\text{PW}_a$ . Then there exists a family of entire functions  $\{P_t\}_{t \in [0, 2a]}$  such that  $\mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^r f(t) P_t(z) dt$  is the unitary operator from  $L^2[0, r]$  to  $(\text{PW}_{[0,r]}, \mu)$  for every  $r \in [0, 2a]$ .*

In the case where  $\mu$  is a “small perturbation” of the Lebesgue measure (see discussion above), the functions  $P_r$  in Corollary 2 coincide with orthogonal entire functions constructed by M. G. Krein in [10]. S. A. Denisov provides an extensive treatment of the subject, collecting many old and new results in paper [5].

The second application of Theorem 1 concerns the classical factorization problem for positive invertible operators. Let  $H$  be a separable Hilbert space and let  $B(H)$  be the algebra of all bounded operators on  $H$ . Consider a complete chain  $\mathcal{N}$  of subspaces in  $H$  and denote by  $\mathcal{A}_\mathcal{N} = \{A \in B(H) : AE \subset E, E \in \mathcal{N}\}$  the nest algebra of upper-triangular operators with respect to  $\mathcal{N}$ . In sixties, I. C. Gohberg and M. G. Krein proved (see Theorem 6.2 in Chapter 4 of [7]) that every positive invertible operator  $T$  on  $H$  of the form  $T = I - K$  with  $K$  in Macaev ideal  $S_\omega$  admits the triangular factorization  $T = A^*A$ , where  $A = I - K_A$  is an invertible operator on  $H$  such that  $K_A \in S_\omega \cap \mathcal{A}_\mathcal{N}$ . Famous theorem by D. R. Larson [11] says that every positive invertible operator  $T$  admits triangular factorization  $T = A^*A$  with  $A, A^{-1} \in \mathcal{A}_\mathcal{N}$  if and only if the chain  $\mathcal{N}$  is countable. Moreover, given  $0 < \varepsilon < 1$ , the non-factorable operator  $T$  can be chosen so that  $K = I - T$  is a compact operator with  $\|K\| < \varepsilon$ .

We consider the problem of triangular factorization for Wiener-Hopf convolution operators. Let  $\psi \in \mathcal{S}'$  be a tempered distribution on  $\mathbb{R}$  and let  $0 < a \leq \infty$ . The Wiener-Hopf operator  $W_\psi$  on  $L^2[0, a)$  with symbol  $\psi$  is densely defined by

$$(W_\psi f)(y) = \langle \psi, s_y f \rangle_{\mathcal{S}'}, \quad y \in [0, a), \quad s_y f : x \mapsto f(x - y),$$

on smooth functions  $f$  with compact support in  $(0, a)$ . In the case where  $\psi \in L^1(\mathbb{R})$  we have more familiar definition,  $W_\psi : f \mapsto \int_0^a \psi(x - y) f(x) dx$ . As following result shows, Wiener-Hopf operators with real symbols are always factorable.

**Theorem 3.** *Let  $0 < a \leq \infty$ . Every positive, bounded, and invertible Wiener-Hopf operator  $W_\psi$  on  $L^2[0, a)$  with real symbol  $\psi \in \mathcal{S}'$  admits triangular factorization:  $W_\psi = A^*A$ , where  $A$  is a bounded invertible operator such that  $AL^2[0, r] = L^2[0, r]$  for every  $r \in [0, a)$ .*

Wiener-Hopf operators  $W_\psi$  in Theorem 3 admit triangular factorizations in the reverse order  $W_\psi = AA^*$  as well. Relation of absolute continuity of aforementioned function  $r \mapsto \|T_{\mu,r}^{-1} \text{sinc}_r\|_{L^2(\mu)}^2$  to triangular factorization problems has been previously found in different terms by L. A. Sakhnovich, see Theorem 4.2 in [16]. On the other hand, Theorem 3 contradicts Theorem 4.1 from another work [17] by the same author. See discussion in Section 5.

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## 2. INTEGRATION OVER SIMPLEX AND THE MUCKENHOUPH CLASS $A_2$

Let  $w$  be a positive function on an interval  $[0, a]$ . We associate to  $w$  the quantity

$$\|w\|_{A_2[0,a]} = \sup_{I \subset [0,a]} \left( \frac{1}{|I|} \int_I w(x) dx \right) \cdot \left( \frac{1}{|I|} \int_I \frac{1}{w(x)} dx \right),$$

where  $I$  runs over all subintervals of  $[0, a]$ . Note that  $\|\cdot\|_{A_2[0,a]}$  is not a norm in the standard sense, but we will use this convenient notation. The Muckenhoupt class  $A_2[0, a]$  consists of functions  $w > 0$  such that  $\|w\|_{A_2[0,a]} < \infty$ . In this section we present a special integral condition for a weight  $w$  to belong to the  $A_2[0, a]$  class.

Let  $\varphi$  be a real-valued function on the interval  $[0, a]$ . For a real  $0 < t < a$  and an integer  $n \geq 1$  define the mapping

$$G_{\varphi,n} : x \mapsto \sum_{k=1}^n (-1)^{n+k} \varphi(x_k), \quad x \in K_{t,n}, \quad (5)$$

on simplex  $K_{t,n} = \{x \in \mathbb{R}^n : x = (x_1, \dots, x_n), t \geq x_1 \geq \dots \geq x_n \geq 0\}$ . Let  $m_n$  denote the usual Lebesgue measure on  $\mathbb{R}^n$ .

Next proposition will be used in the proof of Theorem 1.

**Proposition 2.1.** *Let  $\varphi$  be a function on  $[0, a]$  such that  $e^{|\varphi|} \in L^1[0, a]$ . Assume that for every  $r \in [0, a]$  and every integer  $n \geq 1$  we have*

$$\frac{1}{a_n(r)} \int_0^r e^{(-1)^n \varphi(t)} \left( \int_{K_{t,n}} e^{G_{\varphi,n}(x)} dm_n(x) \right)^2 dt \leq b_2, \quad (6)$$

$$b_1 \leq \frac{1}{r} \int_0^r e^{\varphi(t)} dt \leq b_2; \quad (7)$$

where  $b_1, b_2$  are positive constants, and  $a_n(r) = r^{2n+1}(2n+1)^{-1}(n!)^{-2}$ . Then the function  $w = e^\varphi$  belongs to  $A_2[0, a]$  and  $\|w\|_{A_2[0,a]} \leq 2^{28}(b_2 + b_1^{-2}b_2)^{14}$ .

We first prove several preliminary estimates.

**Lemma 2.1.** *Let  $\varphi$  be a function as in Proposition 2.1. Then for every  $r \in [0, a]$  and  $b = 2(b_2 + b_1^{-2}b_2)$  we have*

$$\frac{1}{r} \int_0^r |\varphi(t)| dt \leq \log b, \quad \frac{1}{r} \int_0^r e^{|\varphi(t)|} dt \leq b. \quad (8)$$

Consequently, for every decreasing differentiable function  $k \geq 0$  on  $[0, r]$  satisfying  $\int_0^r k(t) dt = 1$  and  $k(r) = 0$  we have  $\int_0^r |\varphi(t)| k(t) dt \leq \log b$ .

**Proof.** Clearly, the first estimate in (8) follows from the second one and the Jensen's inequality for convex function  $e^x$ . Taking  $n = 1$  in (6), we obtain

$$\frac{3}{r^3} \int_0^r e^{-\varphi(t)} \left( \int_0^t e^{\varphi(t_1)} dt_1 \right)^2 dt \leq b_2.$$

From (7) we know that  $\frac{1}{t} \int_0^t e^{\varphi(t_1)} dt_1 \geq b_1$  for all  $t \in [0, r]$ . It follows that

$$b_1^{-2} b_2 \geq \frac{3}{r^3} \int_0^r e^{-\varphi(t)} t^2 dt \geq \frac{1}{r} \int_{r/2}^r e^{-\varphi(t)} dt.$$

Using the other side estimate  $\frac{1}{r} \int_0^r e^{\varphi(t)} dt \leq b_2$  and inequality  $e^{|x|} \leq e^x + e^{-x}$ , we see that

$$\frac{2}{r} \int_{r/2}^r e^{|\varphi(t)|} dt \leq b_2 + b_1^{-2} b_2$$

for all  $r \in [0, a]$ . Then (8) follows from

$$\frac{1}{r} \int_0^r e^{|\varphi(t)|} dt = \frac{1}{r} \left( \sum_{k=0}^{\infty} |I_{r,k}| \cdot \frac{1}{|I_{r,k}|} \int_{I_{r,k}} e^{|\varphi(t)|} dt \right) \leq b,$$

where  $I_{r,k} = [2^{-k-1}r, 2^{-k}r]$ . Now if  $k$  is a function on  $[0, r] \subset [0, a]$  as in the statement, we have

$$\begin{aligned} \int_0^r |\varphi(t)| k(t) dt &= - \int_0^r |\varphi(t)| \int_0^t \chi_{[t,r]}(s) k'(s) ds dt \\ &= - \int_0^r k'(s) \int_0^s \chi_{[0,s]}(t) |\varphi(t)| dt ds \\ &\leq - \log b \int_0^r k'(s) s ds = \log b. \end{aligned}$$

This completes the proof.  $\square$

For  $n \geq 1$  introduce the intervals  $I_{t,n} = [\delta_n t, t]$ , where  $\delta_n = 1 - \frac{1}{n+1}$  if  $n$  is odd, and  $\delta_n = 1 - \frac{1}{n}$  if  $n$  is even. In particular,  $I_{t,n} = I_{t,n+1}$  for every odd  $n$ . Set

$$[\varphi]_{t,n} = 2(-1)^{n+1} \int_{K_{t,n}} G_{\varphi,n}(x) dm_{t,n}(x),$$

where  $m_{t,n} = \frac{n!}{t^n} \cdot m_n$  is the scalar multiple of the Lebesgue measure  $m_n$  on  $\mathbb{R}^n$  normalized so that  $m_{t,n}(K_{t,n}) = 1$ .

**Lemma 2.2.** *For  $r \in [0, a]$  and odd  $n \geq 1$  we have  $|[\varphi]_{\delta_n r, n} - [\varphi]_{\delta_{n+1} r, n+1}| < 6 \log b$ , where  $b$  is the constant from Lemma 2.1.*

**Proof.** Arguing by induction, it is easy check that for all  $n \geq 1$  and  $\tau \in [0, a]$  we have

$$[\varphi]_{\tau, n} = \int_0^\tau \varphi(s) k_{\tau, n}(s) ds, \quad k_{\tau, n}(s) = \frac{2n}{\tau^n} (2s - \tau)^{n-1}.$$

For odd (correspondingly, even) integers  $n$  the kernels  $k_{\tau, n}$  are even (correspondingly, odd) functions with respect to the point  $\tau/2$ . As  $n$  tends to infinity, the kernels  $k_{\tau, n}$  tend to zero uniformly on every closed interval in  $(0, \tau)$ . We also have

$$\int_0^\tau |k_{\tau, n}(s)| ds = 2, \quad \sup_{s \in [\tau/2, \tau]} |k_{\tau, n}(s) - k_{\tau, n+1}(s)| \leq \frac{2}{\tau}. \quad (9)$$

Now take an odd integer  $n \geq 1$  and note that  $\delta_n = \delta_{n+1} = \frac{1}{n+1}$ . Setting  $\tau = \delta_n r$ , we obtain

$$\begin{aligned} \left| [\varphi]_{\tau,n} - [\varphi]_{\tau,n+1} \right| &\leq \int_0^{\tau/2} |\varphi(s) k_{\tau,n}(s)| ds \\ &\quad + \int_0^{\tau/2} |\varphi(s) k_{\tau,n+1}(s)| ds \\ &\quad + \int_{\tau/2}^{\tau} |\varphi(s)| \cdot |k_{\tau,n}(s) - k_{\tau,n+1}(s)| ds. \end{aligned}$$

By Lemma 2.1 for functions  $\varphi$ ,  $k = \frac{1}{2}|k_{\tau,n}|$ , and  $k = \frac{1}{2}|k_{\tau,n+1}|$  on  $[0, \frac{\tau}{2}]$ , the sum of first two integrals is bounded from above by  $4 \log b$ . To show that the last integral does not exceed  $2 \log b$ , use (8) and the second estimate in (9).  $\square$

**Proof of Proposition 2.1.** Take an odd integer  $n \geq 1$ . Since the integrand in (6) is positive, we have

$$\begin{aligned} b_2 &\geq \frac{1}{a_n(r)} \int_{\delta_n r}^r e^{-\varphi(t)} \left( \int_{K_{t,n}} e^{G_{\varphi,n}(x)} dm_n(x) \right)^2 dt, \\ &\geq \frac{1}{a_n(r)} \left( \int_{\delta_n r}^r e^{-\varphi(t)} dt \right) \cdot \left( \int_{K_{\delta_n r,n}} e^{G_{\varphi,n}(x)} dm_n(x) \right)^2. \end{aligned}$$

By Jensen's inequality,

$$\int_{K_{\delta_n r,n}} e^{G_{\varphi,n}(x)} dm_n(x) \geq \frac{(\delta_n r)^n}{n!} \exp\left(\frac{[\varphi]_{\delta_n r,n}}{2}\right).$$

For all  $n \geq 1$  we have

$$\frac{1}{a_n(r)} \cdot \left( \frac{(\delta_n r)^n}{n!} \right)^2 = \frac{(2n+1)(n!)^2}{r^{2n+1}} \cdot \frac{r^{2n}}{(n!)^2} \delta_n^{2n} \geq \frac{n+1}{32r} = \frac{1}{32|I_{n,r}|}.$$

We now see that

$$\frac{1}{|I_{r,n}|} \int_{I_{r,n}} \exp(-\varphi(t) + [\varphi]_{\delta_n r,n}) dt \leq 32b_2. \quad (10)$$

Analogously, for the even integer  $n+1$  we have

$$\frac{1}{|I_{r,n+1}|} \int_{I_{r,n+1}} \exp(\varphi(t) - [\varphi]_{\delta_{n+1}r,n}) dt \leq 32b_2.$$

Recall that  $I_{t,n+1} = I_{t,n}$ . Applying Lemma 2.2, we obtain

$$\frac{1}{|I_{r,n}|} \int_{I_{r,n}} \exp(\varphi(t) - [\varphi]_{\delta_n r,n}) dt \leq 32b_2 e^{6 \log b} \leq 32b^7, \quad (11)$$

where  $b$  is the constant from Lemma 2.1. Using inequality  $e^{|x|} \leq e^x + e^{-x}$ , we get from (10) and (11) the estimate

$$\frac{1}{|I|} \int_I e^{|\varphi(t) - c_I|} dt \leq 64b^7 \quad (12)$$

for all intervals  $I$  of the form  $I = [(1 - \frac{1}{n+1})r, r]$ , where  $r \in [0, a]$ , and integer  $n \geq 1$  is odd. Here  $c_I$  is a constant depending on  $I$  (in fact,  $c_I = [\varphi]_{\delta_n r,n}$  works, but from now on the particular choice of  $c_I$  plays no role). Formula (8) gives (12) with  $c_I = 0$  for intervals of the form  $I = [0, t]$ .

Next, observe that each interval  $J \subset [0, a]$  is contained in an interval  $I$  satisfying (12) and such that  $|I| \leq 2|J|$ . Indeed, let  $t$  be the right point of  $J$ . If  $|J| \geq |t|/2$ , take  $I = [0, t]$ . In the case  $|J| < |t|/2$  find an odd number  $n \geq 1$  such that  $I_{t,n+2} \subset J \subset I_{t,n}$  and take  $I = I_{t,n}$ . Fix this interval  $I$  and the corresponding constant  $c_I$  from (12). We have

$$\begin{aligned} \left( \frac{1}{|J|} \int_J e^\varphi dt \right) \cdot \left( \frac{1}{|J|} \int_J e^{-\varphi} dt \right) &\leq \left( \frac{2}{|I|} \int_I e^\varphi dt \right) \cdot \left( \frac{2}{|I|} \int_I e^{-\varphi} dt \right) \\ &\leq \left( \frac{2}{|I|} \int_I e^{\varphi - c_I} dt \right) \cdot \left( \frac{2}{|I|} \int_I e^{-\varphi + c_I} dt \right) \leq (2b)^{14}. \end{aligned}$$

Since interval  $J$  is arbitrary, this shows that function  $w = e^\varphi$  belongs to the Muckenhoupt class  $A_2[0, a]$  and  $\|w\|_{A_2[0, a]} \leq (2b)^{14} = 2^{28}(b_2 + b_1^{-2}b_2)^{14}$ .  $\square$

### 3. PROOF OF THEOREM 1

As it was mentioned in the Introduction, we will use an approximation argument in the proof of Theorem 1. To have a stable approximation, we need a result describing positive truncated Toeplitz operators on  $\text{PW}_a$ .

**3.1. Preliminaries on truncated Toeplitz operators.** Let  $\mu \geq 0$  be a measure on the real line  $\mathbb{R}$  such that  $\|f\|_{L^2(\mu)}^2 \leq c\|f\|_{L^2(\mathbb{R})}^2$  for all functions  $f \in \text{PW}_{[0, a]}$ . Define the truncated Toeplitz operator  $A_{\mu, a}$  on  $\text{PW}_{[0, a]}$  by the sesquilinear form

$$(A_{\mu, a}f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f\bar{g} d\mu, \quad f, g \in \text{PW}_{[0, a]}. \quad (13)$$

In the case where  $\mu = u dm$  is absolutely continuous with respect to the Lebesgue measure  $m$  on  $\mathbb{R}$  and has density  $u$ , the operator  $A_{\mu, a}$  coincides with the projection of the standard Toeplitz operator  $T_u$  on the Hardy space  $H^2$  to the subspace  $\text{PW}_{[0, a]}$ . This explains the name “truncated Toeplitz” for the operator  $A_{\mu, a}$ .

It is well-known (see, e.g., Section 6.1 in [14]) that the operator

$$V : h \mapsto \frac{1}{\sqrt{\pi}} \frac{1}{z+i} h \left( \frac{z-i}{z+i} \right), \quad z \in \mathbb{C}_+, \quad (14)$$

maps unitarily the Hardy space  $H^2(\mathbb{D})$  in the open unit disk  $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$  onto the Hardy space  $H^2$  in the upper half-plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ . Moreover, for every  $a > 0$  we have  $VK_{\theta_a} = \text{PW}_{[0, a]}$ , where  $\theta_a = \exp(a \frac{z+1}{z-1})$  is the inner function in  $\mathbb{D}$  and  $K_{\theta_a}$  is the orthogonal complement in  $H^2(\mathbb{D})$  to the subspace  $\theta_a H^2(\mathbb{D})$ . As we will see in a moment, the truncated Toeplitz operators defined by (13) are unitarily equivalent to truncated Toeplitz operators on the shift-covariant subspace  $K_{\theta_a}$  of  $H^2(\mathbb{D})$ . See D. Sarason’s paper [18] for basic properties of truncated Toeplitz operators on general coinvariant subspaces of  $H^2(\mathbb{D})$ .

We also will deal with the operators  $T_{\mu, a}$  on the space  $\text{PW}_a$  defined by the same sesquilinear form

$$(T_{\mu, a}f, g) = \int_{\mathbb{R}} f\bar{g} d\mu, \quad f, g \in \text{PW}_a.$$

It is easy to see that this definition agrees with formula (4). By construction, we have  $T_{\mu, a} = V_a^{-1} A_{\mu, 2a} V_a$ , where  $V_a : \text{PW}_a \rightarrow \text{PW}_{[0, 2a]}$  is the unitary operator taking a function  $f$  into  $e^{iaz} f$ .

**Lemma 3.1.** *Let  $T$  be a positive bounded operator on  $\text{PW}_{[0,a]}$  satisfying relation*

$$(Tf, f)_{L^2(\mathbb{R})} = (T \frac{z-i}{z+i} f, \frac{z-i}{z+i} f)_{L^2(\mathbb{R})} \quad (15)$$

*for all functions  $f \in \text{PW}_{[0,a]}$  such that  $f(-i) = 0$ . Then there exists a positive measure  $\mu$  on  $\mathbb{R}$  such that  $T = A_{\mu,a}$ . Similarly if  $T$  is a positive bounded operator on  $\text{PW}_a$  satisfying (15) for all  $f \in \text{PW}_a$  such that  $f(-i) = 0$ , then  $T = T_{\mu,a}$  for a positive measure  $\mu$  on  $\mathbb{R}$ .*

**Proof.** Let  $\theta_a$ ,  $K_{\theta_a}$ , and  $V : K_{\theta_a} \rightarrow \text{PW}_{[0,a]}$  be defined as above. Consider the operator  $\tilde{T} = V^{-1}TV$  on  $K_{\theta_a}$  unitarily equivalent to the operator  $T$  on  $\text{PW}_{[0,a]}$ . Recall that the inner product in  $K_{\theta_a}$  is inherited from the space  $L^2(\mathbb{T})$  on the unit circle  $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ . Assumption (15) means that

$$(\tilde{T}h, h)_{L^2(\mathbb{T})} = (\tilde{T}\xi h, \xi h)_{L^2(\mathbb{T})} \quad (16)$$

for every function  $h \in K_{\theta_a}$  such that  $\xi h \in K_{\theta_a}$ . Indeed,  $(V\xi h)(z) = \frac{z-i}{z+i}(Vh)(z)$  and hence  $V(\xi h) \in \text{PW}_{[0,a]}$  if and only if  $(Vh)(-i) = 0$ . Theorem 8.1 in [18] says that a bounded operator  $\tilde{T}$  on  $K_{\theta_a}$  (or on any other coinvariant subspace  $K_{\theta}^2$  of the Hardy space  $H^2(\mathbb{D})$ ) satisfying (16) is a truncated Toeplitz operator on  $K_{\theta_a}$ . By Theorem 2.1 in [1], for every positive bounded truncated Toeplitz operator  $\tilde{T}$  on  $K_{\theta_a}$  there exists a finite positive measure  $\tilde{\mu}$  on  $\mathbb{T}$  such that  $\tilde{\mu}(\{1\}) = 0$  and

$$(\tilde{T}h, h)_{L^2(\mathbb{T})} = \int_{\mathbb{T}} |h|^2 d\tilde{\mu}$$

for all continuous functions  $h$  in  $K_{\theta_a}$ . Changing variables in the last integral, we find a positive measure  $\mu$  on  $\mathbb{R}$  such that

$$\int_{\mathbb{T}} |h|^2 d\tilde{\mu} = \int_{\mathbb{R}} |f|^2 d\mu, \quad f = Vh.$$

It follows that  $(Tf, f) = (\tilde{T}h, h)_{L^2(\mathbb{T})} = (A_{\mu,a}f, f)_{L^2(\mathbb{R})}$  for a dense set of functions  $f$  in  $\text{PW}_{[0,a]}$ . Since  $T$  is continuous, we have  $T = A_{\mu,a}$ . The second part of the Lemma is a direct consequence of relation  $T_{\mu,a} = V_a^{-1}A_{\mu,2a}V_a$ .  $\square$

**3.2. Preliminaries on canonical Hamiltonian systems.** Let  $\mathcal{H}$  be a Hamiltonian on  $[0, a]$  with trace  $\mathcal{H} \in L^1[0, a]$ . Assume that there is no interval  $(r_1, r_2) \subset [0, a]$  such that  $\mathcal{H}(t)$  is a constant matrix of rank one for all points  $t \in (r_1, r_2)$ . For  $r \in [0, a]$  we will denote by  $\mathcal{B}(\mathcal{H}, r)$  the de Branges space generated by  $\mathcal{H}$  on  $[0, r]$ , that is,

$$\mathcal{B}(\mathcal{H}, r) = \mathcal{W}_{\mathcal{H},r} L^2(\mathcal{H}, r) = \left\{ \text{entire } f : f = \mathcal{W}_{\mathcal{H},r} X, \quad X \in L^2(\mathcal{H}, r) \right\},$$

where the Weyl-Titchmarsh transform  $\mathcal{W}_{\mathcal{H},r}$  is defined in (3) for  $a = r$ . The space  $\mathcal{B}(\mathcal{H}, r)$  is actually the Hilbert space with respect to the inner product  $(f, g)_{\mathcal{B}(\mathcal{H}, r)} = (f, g)_{L^2(\mu)}$ , where  $\mu$  is any spectral measure for problem (2). We refer the reader to paper [2] for the summary of results on direct and inverse spectral theory of canonical Hamiltonian systems and de Brange spaces of entire functions. The readers interested in proofs or in a more detailed account may find necessary information in Chapter 2 of classical book [4] by L. de Brange or its recent exposition [15] by R. Romanov.



**Lemma 3.2.** *Let  $\mu$  be an even measure on  $\mathbb{R}$  of the form  $\mu = c\mathfrak{m} + \nu$ , where  $c > 0$  and  $\nu$  is a finite positive measure on  $\mathbb{R}$  with compact support. Then there exists an infinitely smooth diagonal Hamiltonian  $\mathcal{H}$  on  $[0, +\infty)$  such that  $\det \mathcal{H}(r) = 1$  for all  $r \geq 0$ , and  $\mu$  is the spectral measure for  $\mathcal{H}$ .*

**Proof.** The result is a kind of folklore. Since the Fourier transform of  $\frac{1}{c}\nu$  is a smooth (in fact, analytic) function, one can use the classical Gelfand-Levitan approach to find a smooth diagonal potential  $Q$  on  $[0, a]$  such that  $m + \frac{1}{c}\nu$  is the spectral measure for the Dirac system  $JY' + QY = zY$  corresponding to the boundary condition  $Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Then rewrite system  $JY' + QY = zY$  as a canonical Hamiltonian system  $JX' = z\tilde{\mathcal{H}}X$  setting  $X = M^{-1}Y$ ,  $\tilde{\mathcal{H}} = M^*M$ , where  $M$  is the matrix solution of equation  $JM' = -QM$ ,  $M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Observe that  $\det \tilde{\mathcal{H}} = 1$  almost everywhere on  $[0, a]$  and  $m + \frac{1}{c}\nu$  is the spectral measure for system  $JX' = z\tilde{\mathcal{H}}X$ ,  $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . To obtain the Hamiltonian on  $[0, a]$  corresponding to the spectral measure  $\mu$ , put  $\mathcal{H} = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \tilde{\mathcal{H}}$ . Another (in a sense, equivalent) way of proving Lemma 3.2 is the application of Theorem 5.1 from [20].  $\square$

Define  $\text{type } \mathcal{B}(\mathcal{H}, r) = \sup\{\text{type}(f), f \in \mathcal{B}(\mathcal{H}, r)\}$  to be the maximal exponential type of entire functions in de Branges space  $\mathcal{B}(\mathcal{H}, r)$ . The following remarkable formula of Krein [9] and de Brange (Theorem X in [3])

$$\text{type } \mathcal{B}(\mathcal{H}, r) = \int_0^r \sqrt{\det \mathcal{H}(t)} dt, \quad (17)$$

represents the maximal exponential type of functions in  $\mathcal{B}(\mathcal{H}, r)$  in terms of the Hamiltonian  $\mathcal{H}$ . Section 6 in [15] contains an elegant self-contained proof of this result.

**Lemma 3.3.** *Let  $\mathcal{H}$  be a Hamiltonian on an interval  $[0, a]$  such that its spectral measure  $\mu$  satisfies (1). Assume that  $\det \mathcal{H}(r) = 1$  for almost all  $r \in [0, a]$ . Then for all  $r \in [0, a]$  we have  $\mathcal{B}(\mathcal{H}, r) = (\text{PW}_r, \mu)$ .*

**Proof.** Let  $r \in [0, a)$  and let  $\varepsilon > 0$  be such that  $r \in [\varepsilon, a - \varepsilon)$ . Then the Hilbert space  $(\text{PW}_{r+\varepsilon}, \mu)$  of entire functions satisfies an axiomatic description of de Branges spaces (Theorem 23 in [4]) and the embedding  $(\text{PW}_{r+\varepsilon}, \mu) \subset L^2(\mu)$  is isometric. Since  $\mu$  is a spectral measure for  $\mathcal{H}$ , the embedding  $\mathcal{B}(\mathcal{H}, r) \subset L^2(\mu)$  is isometric as well. Applying de Branges chain theorem (Theorem 35 in [4]), we see that either  $(\text{PW}_{r+\varepsilon}, \mu) \subset \mathcal{B}(\mathcal{H}, r)$  or  $\mathcal{B}(\mathcal{H}, r) \subset (\text{PW}_{r+\varepsilon}, \mu)$ . Since  $\det \mathcal{H} = 1$  almost everywhere on  $[0, a]$ , formula (17) implies the second alternative. Analogously, one can show that  $(\text{PW}_{r-\varepsilon}, \mu) \subset \mathcal{B}(\mathcal{H}, r)$ . Since this holds for every small number  $\varepsilon$  and  $\mu$  is sampling, we have  $\mathcal{B}(\mathcal{H}, r) = (\text{PW}_r, \mu)$ . Finally, for  $r = a$  we have

$$\mathcal{B}(\mathcal{H}, a) = \overline{\bigcup_{0 < r < a} \mathcal{B}(\mathcal{H}, r)} = (\text{PW}_a, \mu),$$

where the completion is taken with respect to the norm inherited from  $L^2(\mu)$ .  $\square$

Let  $\Theta_{\mathcal{H}}$  be the absolutely continuous solution of Cauchy problem (2) on  $[0, a]$ , and denote  $\Theta_{\mathcal{H}}^+ = \langle \Theta_{\mathcal{H}}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ ,  $\Theta_{\mathcal{H}}^- = \langle \Theta_{\mathcal{H}}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$ . The reproducing kernel  $k_{\mathcal{B}(\mathcal{H}, r); \lambda}$  at a point  $\lambda \in \mathbb{C}$  of the Hilbert space of entire functions  $\mathcal{B}(\mathcal{H}, r)$  has the form

$$k_{\mathcal{B}(\mathcal{H}, r); \lambda} = \frac{1}{\pi} \frac{\Theta_{\mathcal{H}}^+(r, z)\Theta_{\mathcal{H}}^-(r, \bar{\lambda}) - \Theta_{\mathcal{H}}^-(r, z)\Theta_{\mathcal{H}}^+(r, \bar{\lambda})}{z - \bar{\lambda}}, \quad z \in \mathbb{C}. \quad (18)$$

The Paley-Wiener space  $\text{PW}_r$  is the de Branges space  $\mathcal{B}(\mathcal{H}_0, r)$  for the Hamiltonian  $\mathcal{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The reproducing kernel of  $\text{PW}_r$  at  $\lambda \in \mathbb{C}$  will be denoted by  $\text{sinc}_{r,\lambda}$ :

$$\text{sinc}_{r,\lambda} = \frac{\sin r(z - \bar{\lambda})}{\pi(z - \bar{\lambda})}, \quad z \in \mathbb{C}.$$

Using integration by parts and equation (2), it is easy to show that for each  $\lambda \in \mathbb{C}$  we have

$$\mathcal{W}_{\mathcal{H},r} \Theta_{\mathcal{H}}(\cdot, \bar{\lambda}) = \sqrt{\pi} k_{\mathcal{B}(\mathcal{H},r);\lambda}, \quad \mathcal{W}_{\mathcal{H}_0,r} \Theta_{\mathcal{H}_0}(\cdot, \bar{\lambda}) = \sqrt{\pi} \text{sinc}_{r,\lambda},$$

where  $\Theta_{\mathcal{H}}(\cdot, \bar{\lambda})$  denotes the mapping  $t \mapsto \Theta_{\mathcal{H}}(t, \bar{\lambda})$  and  $\Theta_{\mathcal{H}_0}(\cdot, \bar{\lambda})$  is defined analogously.

Next assertion is Lemma 4.2 in [2].

**Lemma 3.4.** *Let  $\mu$  be a sampling measure for  $\text{PW}_a$  and let  $r \in [0, a]$ . The reproducing kernel of the space  $(\text{PW}_r, \mu)$  at  $\lambda \in \mathbb{C}$  equals  $T_{\mu,r}^{-1} \text{sinc}_{r,\lambda}$ .*

**Proof.** For every function  $f$  in  $(\text{PW}_r, \mu) \subset \text{PW}_r$  and every  $\lambda \in \mathbb{C}$  we have

$$f(\lambda) = (f, \text{sinc}_{a,\lambda})_{L^2(\mathbb{R})} = (f, T_{\mu,r}^{-1} \text{sinc}_{r,\lambda})_{L^2(\mu)},$$

where we used the fact that  $c_1 I \leq T_{\mu,a} \leq c_2 I$  on  $\text{PW}_a$  and hence  $T_{\mu,r}$  is bounded and invertible on  $\text{PW}_r$ .  $\square$

**Lemma 3.5.** *Let  $\varphi$  be a function on  $[0, a]$  such that  $e^{|\varphi|} \in L^1[0, a]$ . Assume that a spectral measure  $\mu$  of problem (2) for the canonical Hamiltonian system generated by  $\mathcal{H} = \begin{pmatrix} e^\varphi & 0 \\ 0 & e^{-\varphi} \end{pmatrix}$  satisfies (1) for some constants  $c_1, c_2$ . Then function  $w = e^\varphi$  belongs to the Muckenhoupt class  $A_2[0, a]$  and  $\|w\|_{A_2[0,2]} \leq 2^{28} c^{14}$ , where  $c = c_1^{-1} + c_2^2 c_1^{-1}$ . We also have  $\frac{1}{a} \int_0^a (w + \frac{1}{w}) dx \leq 4c$ .*

**Proof.** Let us obtain estimates (6), (7) for the function  $\varphi$  as it was suggested in Proposition 3.2 of [2]. Take  $r \in [0, a]$ . Set  $\mathcal{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and consider the corresponding Weyl-Titchmarsh transforms

$$\mathcal{W}_{\mathcal{H}_0,r} : L^2(\mathcal{H}_0, r) \rightarrow \mathcal{B}(\mathcal{H}_0, r), \quad \mathcal{W}_{\mathcal{H},r} : L^2(\mathcal{H}, r) \rightarrow \mathcal{B}(\mathcal{H}, r).$$

We have  $\mathcal{B}(\mathcal{H}_0, r) = \text{PW}_r$  and  $\mathcal{B}(\mathcal{H}, r) = (\text{PW}_r, \mu)$ , see Lemma 3.3. Since  $\mu$  satisfies (1), the spaces  $\text{PW}_r, (\text{PW}_r, \mu)$  coincide as sets and

$$c_2^{-1} \|f\|_{L^2(\mathbb{R})}^2 \leq \|T_{\mu,r}^{-1} f\|_{L^2(\mu)}^2 \leq c_1^{-1} \|f\|_{L^2(\mathbb{R})}^2$$

for every function  $f \in \text{PW}_r$ . Hence, the operator  $T = \mathcal{W}_{\mathcal{H},r}^{-1} T_{\mu,r}^{-1} \mathcal{W}_{\mathcal{H}_0,r}$  from  $L^2(\mathcal{H}_0, r)$  to  $L^2(\mathcal{H}, r)$  is correctly defined, bounded, and invertible. Moreover,

$$c_2^{-1} \|X\|_{L^2(\mathcal{H}_0,r)}^2 \leq \|TX\|_{L^2(\mathcal{H},r)}^2 \leq c_1^{-1} \|X\|_{L^2(\mathcal{H}_0,r)}^2 \quad (19)$$

for every  $X \in L^2(\mathcal{H}_0)$ . Next, by Lemma 3.4 for each  $z \in \mathbb{C}$  we have

$$T \Theta_{\mathcal{H}_0}(\cdot, z) = \mathcal{W}_{\mathcal{H},r}^{-1} (\sqrt{\pi} T_{\mu,r}^{-1} \text{sinc}_{r,\bar{z}}) = \Theta_{\mathcal{H}}(\cdot, z).$$

For  $z = 0$  and all  $t \in [0, r]$  we have  $\Theta_{\mathcal{H}}(t, 0) = \Theta_{\mathcal{H}_0}(t, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , hence

$$c_2^{-1} \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{L^2(\mathcal{H}_0,r)}^2 \leq \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{L^2(\mathcal{H},r)}^2 \leq c_1^{-1} \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{L^2(\mathcal{H}_0,r)}^2.$$

This relation is inequality (7) for the function  $\varphi$  and constants  $b_1 = c_2^{-1}, b_2 = c_1^{-1}$ .

Now let  $\partial_0^n \Theta_{\mathcal{H}}(\cdot, 0)$  denote the derivative of order  $n$  of the mapping  $z \mapsto \Theta_{\mathcal{H}}(\cdot, z)$  from  $\mathbb{C}$  to  $L^2(\mathcal{H}, r)$  at the point  $z = 0$ . Then  $T\partial_0^n \Theta_{\mathcal{H}}(\cdot, 0) = \partial_0^n \Theta_{\mathcal{H}_0}(\cdot, 0)$  for all integers  $n \geq 1$ . The right inequality in (19) yields

$$\|\partial_0^n \Theta_{\mathcal{H}}(\cdot, 0)\|_{L^2(\mathcal{H}, r)}^2 \leq c_1^{-1} \|\partial_0^n \Theta_{\mathcal{H}_0}(\cdot, 0)\|_{L^2(\mathcal{H}_0, r)}^2. \quad (20)$$

From equation (2) we obtain

$$\partial_0^n \Theta_{\mathcal{H}}(t, 0) = n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} J^* \mathcal{H}(t_1) J^* \mathcal{H}(t_2) \dots J^* \mathcal{H}(t_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt_n \dots dt_1, \quad (21)$$

$$\partial_0^n \Theta_{\mathcal{H}_0}(t, 0) = J^{*n} \begin{pmatrix} t^n \\ 0 \end{pmatrix}, \quad (22)$$

for all  $t \in [0, r]$  and  $n \geq 1$ . Observe that

$$J^* \mathcal{H}(t_1) J^* \mathcal{H}(t_2) \dots J^* \mathcal{H}(t_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ (-1)^{\frac{n+3}{2}} \exp(G_{\varphi, n}(t)) \end{pmatrix}, & n \text{ is odd,} \\ \begin{pmatrix} (-1)^{\frac{n}{2}} \exp(G_{\varphi, n}(t)) \\ 0 \end{pmatrix}, & n \text{ is even,} \end{cases}$$

where  $t = (t_1, \dots, t_n)$  is a point in simplex  $K_{t, n}$ , and  $G_{\varphi, n}$  is defined on  $K_{t, n}$  by formula (5). Substitute this representation of  $J^* \mathcal{H}(t_1) J^* \mathcal{H}(t_2) \dots J^* \mathcal{H}(t_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  to (21). Then (21), (22), and (20) give us inequality (6) for all  $n \geq 1$  and all  $r \in [0, a]$ . It remains to use Proposition 2.1 to see that  $w \in A_2[0, a]$  and  $\|w\|_{A_2[0, a]} \leq 2^{28} c^{14}$ . The estimate  $\frac{1}{a} \int_0^a (w + \frac{1}{w}) dx \leq 4c$  follows from Lemma 2.1.  $\square$

**3.3. Proof of Theorem 1.** Let  $\mu$  be a measure on  $\mathbb{R}$  such that estimate (1) holds for some  $a > 0$ . Consider the truncated Toeplitz operator  $T_{\mu, a} = T_{\mu}$  on  $\text{PW}_a$ . We have  $c_1 I \leq T_{\mu} \leq c_2 I$ , where  $I$  stands for the identity operator on  $\text{PW}_a$ . The operator  $T_{\mu} - c_1 I$  satisfies assumptions of Lemma 3.1. Hence, there exists a measure  $\nu \geq 0$  on  $\mathbb{R}$  such that  $T_{\nu} = T_{\mu} - c_1 I$ . One can suppose that  $\nu$  is even (otherwise consider the measure  $\tilde{\nu}$  such that  $\tilde{\nu}(S) = \frac{1}{2}(\nu(S) + \nu(-S))$ , and note that  $T_{\nu} = T_{\tilde{\nu}}$ ). Define a sequence of measures  $\mu_j$  by  $\mu_j = c_1 m + \chi_j \nu$ , where  $m$  is the Lebesgue measure on  $\mathbb{R}$ , and  $\chi_j$  denotes the indicator function of the interval  $[-j, j]$ . For every  $j \geq 1$  the measure  $\mu_j$  is even and satisfies relation (1) with the same constants  $c_1, c_2$ . Indeed,  $T_{\mu_j} = c_1 I + T_{\chi_j \nu}$  and

$$c_1 I \leq c_1 I + T_{\chi_j \nu} \leq c_1 I + T_{\nu} = T_{\mu} \leq c_2 I.$$

By Lemma 3.2 and Lemma 3.5, for every  $j$  there exists a smooth function  $w_j > 0$  on the interval  $[0, a]$  such that  $\|w_j\|_{A_2[0, a]} \leq 2^{28} c^{14}$ ,  $c = c_1^{-1} + c_2^2 c_1^{-1}$ , and  $\mu_j$  is the spectral measure for the Hamiltonian  $\mathcal{H}_j = \begin{pmatrix} w_j & 0 \\ 0 & \frac{1}{w_j} \end{pmatrix}$  on  $[0, a]$ . We also have  $\frac{1}{a} \int_0^a (w + \frac{1}{w}) dx \leq 4c$  for all  $j \geq 1$ . This allows us to use “a reverse Hölder inequality” for weights in  $A_2[0, a]$ . It says that for every  $C_1 > 0$  there exist  $p > 1$  and  $C_2 > 0$  such that for all  $h \in A_2[0, a]$  with  $\|h\|_{A_2[0, a]} \leq C_1$  we have

$$\frac{1}{a} \int_0^a h(x)^p dx \leq C_2 \left( \frac{1}{a} \int_0^a h(x) dx \right)^p.$$

Explicit relations between  $C_1, C_2$ , and  $p$  can be found in [19]. From here we see that sequences  $\{w_j\}_{j \geq 1}, \{\frac{1}{w_j}\}_{j \geq 1}$  are infirmly bounded in  $L^p[0, a]$  for some  $p > 1$ . Hence we can find subsequences  $w_{j_k}, w_{j_k}^{-1}$  converging weakly in  $L^p[0, a]$  to functions  $w, v$ , correspondingly. To simplify notations, let the sequences  $\{w_j\}_{j \geq 1}, \{\frac{1}{w_j}\}_{j \geq 1}$  themselves be weakly convergent. Let us show that  $v = w^{-1}$  almost everywhere

on the interval  $[0, a]$ . This is not always the case for arbitrary weakly convergent sequences in  $L^p[0, a]$ .

For  $z \in \mathbb{C}$  denote by  $\Theta_j(\cdot, z)$  solution of equation (2) for the Hamiltonian  $\mathcal{H}_j$ . Integrating (2), we get

$$J\Theta_j(r, z) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = z \int_0^r \mathcal{H}_j(t) \Theta_j(t, z) dt. \quad (23)$$

Then for every  $j \geq 1$  and  $r, r' \in [0, a]$  we have the estimates

$$\begin{aligned} \|\Theta_j(r, z)\|_{\mathbb{C}^2} &\leq \exp\left(|z| \int_0^a \|\mathcal{H}_j(t)\| dt\right), \\ \|\Theta_j(r, z) - \Theta_j(r', z)\|_{\mathbb{C}^2} &\leq |z| \cdot |r - r'|^{\frac{p-1}{p}} \left(\int_0^a \|\mathcal{H}_j(t)\|^p \cdot \|\Theta_j(t, z)\|_{\mathbb{C}^2}^p dt\right)^{\frac{1}{p}}, \end{aligned}$$

showing that functions  $\Theta_j(\cdot, z)$  are uniformly bounded and equicontinuous on  $[0, a]$ . Therefore, there is a subsequence of the sequence  $\Theta_j(\cdot, z)$  converging uniformly on  $[0, a]$  to a function  $\Theta(\cdot, z)$ . As before, we suppose that the sequence  $\Theta_j(\cdot, z)$  itself is uniformly convergent on  $[0, a]$ . It is clear that the limit function  $\Theta$  satisfies equation (23) for the Hamiltonian  $\mathcal{H} = \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}$ . Hence, it satisfies equation (2) for  $\mathcal{H}$ . Fix a number  $r \in (0, a]$ . For every  $\lambda$  and  $z$  in  $\mathbb{C}$  we have

$$k_{\mathcal{B}(\mathcal{H}, r); \lambda}(z) = \lim_{j \rightarrow \infty} k_{\mathcal{B}(\mathcal{H}_j, r); \lambda}(z) = \lim_{j \rightarrow \infty} (T_{\mu_j, r}^{-1} \text{sinc}_{r, \lambda})(z) = (T_{\mu, r}^{-1} \text{sinc}_{r, \lambda})(z). \quad (24)$$

Indeed, the first equality above follows from formula (18) and convergence of  $\Theta_j$  to  $\Theta$  on  $[0, a]$  when a spectral parameter ( $\bar{\lambda}$  or  $z$ ) is fixed. Lemma 3.3 and Lemma 3.4 give us the second equality. Finally, using the fact that the operators  $T_{\mu_j, r}$  on  $\text{PW}_r$  tend to  $T_{\mu, r}$  in the strong operator topology, we obtain the last equality in (24). From (24) we see that Hilbert spaces of entire functions  $\mathcal{B}(\mathcal{H}, r)$ ,  $(\text{PW}_r, \mu)$  have the same reproducing kernels. Hence  $\mathcal{B}(\mathcal{H}, r) = (\text{PW}_r, \mu)$  and formula (17) implies

$$r = \int_0^r \sqrt{\det \mathcal{H}(t)} dt, \quad r \in [0, a].$$

It follows that  $\det \mathcal{H} = 1$  almost everywhere on  $[0, a]$ , that is,  $v = w^{-1}$ . Next, from the direct spectral theory we know that the family  $\{\Theta(\cdot, \lambda)\}_{\lambda \in \mathbb{C}}$  is complete in  $L^2(\mathcal{H}, a)$  and  $\mathcal{W}_{\mathcal{H}, a} \Theta(\cdot, \lambda) = k_{\mathcal{B}(\mathcal{H}, a); \lambda}$  for every  $\lambda \in \mathbb{C}$ , where  $\mathcal{W}_{\mathcal{H}, a}$  denotes the Weyl-Titchmarsh transform associated to  $\mathcal{H}$ . Using (24) again, we get

$$\begin{aligned} (\Theta(\cdot, \lambda), \Theta(\cdot, z))_{L^2(\mathcal{H}, a)} &= \pi k_{\mathcal{B}(\mathcal{H}, a); \lambda}(z) = \pi (T_{\mu, a}^{-1} \text{sinc}_{a, \lambda}, \text{sinc}_{a, z})_{L^2(\mathbb{R})} \\ &= \pi (T_{\mu, a}^{-1} \text{sinc}_{a, \lambda}, T_{\mu, a}^{-1} \text{sinc}_{a, z})_{L^2(\mu)} = (\mathcal{W}_{\mathcal{H}, a} \Theta(\cdot, \lambda), \mathcal{W}_{\mathcal{H}, a} \Theta(\cdot, z))_{L^2(\mu)}. \end{aligned}$$

Hence, the operator  $\mathcal{W}_{\mathcal{H}, a}$  acts isometrically from  $L^2(\mathcal{H}; a)$  to  $L^2(\mu)$  and  $\mu$  is a spectral measure for  $\mathcal{H}$ . In particular, we can apply Lemma 3.5 to  $\mathcal{H}$ ,  $\mu$ , and conclude that the function  $w = e^\varphi$  is in  $A_2[0, a]$  and  $\|w\|_{A_2[0, a]} \leq 2^{28} c^{14}$ . Uniqueness of the Hamiltonian  $\mathcal{H}$  follows immediately from formula (24):

$$\int_0^r w(t) dt = \int_0^r \langle \mathcal{H}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle dt = \pi k_{\mathcal{B}(\mathcal{H}, a); 0}(0) = \pi \|T_{\mu, r}^{-1} \text{sinc}_{r, 0}\|_{L^2(\mu)}^2, \quad (25)$$

where the right hand side is completely determined by  $\mu$ , while the left hand side determines  $\mathcal{H}$ .  $\square$

Differentiating formula (25), we obtain the following corollary.

**Corollary 1.** *The Hamiltonian  $\mathcal{H} = \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix}$  in Theorem 1 could be recovered from  $\mu$  by means of the following formula:  $w(r) = \pi \frac{\partial}{\partial r} \|T_{\mu,r}^{-1} \text{sinc}_{r,0}\|_{L^2(\mu)}^2$ ,  $r \in [0, a]$ .*

#### 4. PROOF OF THEOREM 2 AND THEOREM 3

Let us first show that Theorem 2 does not follow from a general theory of canonical Hamiltonian systems. Consider the simplest case where the Hamiltonian  $\mathcal{H}$  coincides with the identity matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  on  $[0, a]$ . We claim that there is no unitary operator  $U : L^2(\mathcal{H}, a) \rightarrow \text{PW}_{[0,2a]}$  such that  $UL^2(\mathcal{H}, r) = \text{PW}_{[0,2r]}$  for all  $r \in [0, a]$ . Indeed, existence of such a unitary operator yields the existence of another unitary operator  $\tilde{U} : L^2[-a, a] \rightarrow L^2[0, 2a]$  such that  $\tilde{U}L^2[-r, r] = L^2[0, 2r]$  for all  $r \in [0, a]$ . For every  $r_1 > r_2 \geq 0$  let  $\chi_{[r_1, r_2]}$  denote the indicator function of the interval  $[r_1, r_2]$ . Put  $g = \tilde{U}\chi_{[0,a]}$  and consider decomposition  $g = f_r + h_r$ , where  $f_r = \tilde{U}\chi_{[0,r]}$ ,  $h_r = \tilde{U}\chi_{[r,a]}$ ,  $r \in [0, a]$ . Since  $\tilde{U}L^2[-r, r] = L^2[0, 2r]$  by our assumption, the function  $f_r$  is supported on  $[0, 2r]$ . Note also that the function  $h_r$  is orthogonal to all functions from  $L^2[0, 2r]$  and hence it is supported on  $[2r, 2a]$ . From here we see that  $f_r = \chi_{[0,2r]}g$  for all  $r \in [0, a]$ . Next, unitarity of the operator  $\tilde{U}$  implies that

$$\int_0^{2r} |g(t)|^2 dt = \int_0^{2r} |f_r(t)|^2 dt = \int_{-a}^a |\chi_{[0,r]}(t)|^2 dt = r, \quad r \in [0, a].$$

It follows that  $|g(t)|^2 = 1/2$  for almost all  $t \in [0, 2a]$ . In particular, the linear span of functions  $f_r \in \tilde{U}L^2[0, a]$ ,  $r \in [0, a]$ , is dense in  $L^2[0, 2a]$ . This contradicts to the fact that  $\tilde{U}$  is a unitary operator from  $L^2[-a, a]$  to  $L^2[0, 2a]$ . Thus, the Weyl-Titchmarsh transform  $\mathcal{W}_{\mathcal{H},a}$  from formula (3) can not be used to construct the operator  $\mathcal{F}_\mu$  from Theorem 2 by means of superposition with some simple unitary operators like shifts, reflections, etc.

The main point that helps in proof of Theorem 2 is the fact that Hamiltonian  $\mathcal{H}$  generated by an even sampling measure for the Paley-Wiener space  $\text{PW}_a$  must have rank two almost everywhere on its domain of definition. It is an open question if this is true for general (not necessarily even) sampling measures for  $\text{PW}_a$ . See also Proposition 5.1 in Section 5 for more details.

**Proof of Theorem 2.** Fix an even sampling measure  $\mu$  and construct the Hamiltonians  $\mathcal{H}_j$ ,  $\mathcal{H}$ , on  $[0, a]$  as in the proof of Theorem 1. Put  $\varphi_j = \log w_j$  and  $\varphi = \log w$ , where  $w_j$ ,  $w$  are the functions generating  $\mathcal{H}_j$ ,  $\mathcal{H}$ . Recall that  $w_j$  tend to  $w$  weakly in  $L^p[0, a]$  for some  $p > 1$  and the same is true for  $w_j^{-1}$  and  $w^{-1}$ . Let  $\Theta_j$ ,  $\Theta$  be the solutions of system (2) generated by Hamiltonians  $\mathcal{H}_j$ ,  $\mathcal{H}$ , correspondingly. As we have seen, the functions  $\Theta_j(\cdot, z) = \begin{pmatrix} \Theta_j^+ \\ \Theta_j^- \end{pmatrix}$  converge uniformly to  $\Theta(\cdot, z) = \begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix}$  on the interval  $[0, a]$  when  $z \in \mathbb{C}$  is fixed. For  $r \in [0, a]$ , define entire functions  $P_{2r,j}$

and  $P_{2r,j}^*$  by

$$\begin{aligned} P_{2r,j} &: z \mapsto e^{irz} \left( e^{\frac{\varphi_j(r)}{2}} \Theta_j^+(r, z) - i e^{-\frac{\varphi_j(r)}{2}} \Theta_j^-(r, z) \right), \\ P_{2r,j}^* &: z \mapsto e^{irz} \left( e^{\frac{\varphi_j(r)}{2}} \Theta_j^+(r, z) + i e^{-\frac{\varphi_j(r)}{2}} \Theta_j^-(r, z) \right), \end{aligned}$$

and let  $P_{2r}, P_{2r}^*$  be defined similarly with  $\varphi_j$  replaced by  $\varphi$ . These functions satisfy the Krein system of differential equations:

$$\begin{cases} P_{r,j}'(z) = izP_{r,j}(z) + \frac{\varphi_j'(r/2)}{4} P_{r,j}^*(z), & P_{0,j}(z) = e^{\frac{\varphi_j(0)}{2}}, \\ P_{r,j}^{*'}(z) = \frac{\varphi_j'(r/2)}{4} P_{r,j}(z), & P_{0,j}^*(z) = e^{-\frac{\varphi_j(0)}{2}}, \end{cases} \quad (26)$$

where  $\varphi_j'(r/2)$  is the value of smooth function  $\varphi_j'$  at  $r/2$ . From system (26) we obtain by integration by parts (see Lemma 9.1 in [5]) the Christoffel-Darboux formula:

$$\int_0^r P_{t,j}(z) \overline{P_{t,j}(\lambda)} dt = i \frac{P_{r,j}^*(z) \overline{P_{r,j}^*(\lambda)} - P_{r,j}(z) \overline{P_{r,j}(\lambda)}}{z - \bar{\lambda}}.$$

The right hand side could be rewritten in the form

$$\dots = 2e^{i\frac{r}{2}(z-\bar{\lambda})} \cdot \frac{\Theta_j^+(\frac{r}{2}, z) \overline{\Theta_j^-(\frac{r}{2}, \lambda)} - \Theta_j^-(\frac{r}{2}, z) \overline{\Theta_j^+(\frac{r}{2}, \lambda)}}{z - \bar{\lambda}},$$

which tends to  $2\pi k_{r,\lambda}(z)$ , the scalar multiple of the reproducing kernel  $k_{r,\lambda}$  at  $\lambda$  of the Hilbert space  $e^{i\frac{r}{2}z} \mathcal{B}(\mathcal{H}, \frac{r}{2}) = (\text{PW}_{[0,r]}, \mu)$ , see formula (18). On the other hand, for every pair  $z, \lambda \in \mathbb{C}$  we have

$$\begin{aligned} P_{t,j}(z) \overline{P_{t,j}(\lambda)} &= e^{i\frac{t}{2}(z-\bar{\lambda})} \left( e^{\varphi_j(\frac{t}{2})} \Theta_j^+(\frac{t}{2}, z) \overline{\Theta_j^+(\frac{t}{2}, \lambda)} + e^{-\varphi_j(\frac{t}{2})} \Theta_j^-(\frac{t}{2}, z) \overline{\Theta_j^-(\frac{t}{2}, \lambda)} \right. \\ &\quad \left. + i \Theta_j^+(\frac{t}{2}, z) \overline{\Theta_j^-(\frac{t}{2}, \lambda)} - i \Theta_j^-(\frac{t}{2}, z) \overline{\Theta_j^+(\frac{t}{2}, \lambda)} \right). \end{aligned}$$

Since functions  $e^{\varphi_j}, e^{-\varphi_j}$  converge weakly in  $L^p[0, a]$  to functions  $e^\varphi, e^{-\varphi}$ , correspondingly, we see that

$$\int_0^r P_t(z) \overline{P_t(\lambda)} dt = \lim_{j \rightarrow \infty} \int_0^r P_{t,j}(z) \overline{P_{t,j}(\lambda)} dt = 2\pi k_{r,\lambda}(z) \quad (27)$$

for every  $r \in [0, 2a]$ . Let  $\chi_r$  be the indicator function of the interval  $[0, r]$ . Denote by  $L$  the set of all finite linear combinations of functions  $t \mapsto \chi_r(t) \overline{P_t(z)}$  on  $[0, 2a]$ , where  $z \in \mathbb{C}$  and  $r \in [0, 2a]$ . The linear manifold  $L$  is dense in  $L^2[0, 2a]$ . Indeed, for every function  $g \in L^2[0, 2a]$  orthogonal to  $L$  we have

$$0 = \int_0^{2a} g(t) \chi_r(t) P_t(0) dt = \int_0^r g(t) e^{\frac{\varphi(t/2)}{2}} dt, \quad r \in [0, 2a],$$

yielding  $g = 0$  in  $L^2[0, 2a]$ . Formula (27) also shows that a nontrivial finite linear combination of functions  $\chi_r(t) \overline{P_t(z)}$  cannot vanish almost everywhere on  $[0, 2a]$ . Consider the operator  $\mathcal{F}_\mu : L^2[0, 2a] \rightarrow (\text{PW}_{[0,2a]}, \mu)$  densely defined on  $L$  by

$$\mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^{2a} f(t) P_t(z) dt, \quad z \in \mathbb{C}.$$

The operator  $\mathcal{F}_\mu$  takes the function  $t \mapsto \chi_r(t) \overline{P_t(\lambda)}$  on  $[0, 2a]$  into  $\sqrt{2\pi} k_{r,\lambda}$ , see formula (27). Moreover, for every  $r_1, r_2 \in [0, 2a]$  we have

$$\begin{aligned} \left( \mathcal{F}_\mu \chi_{r_1} \overline{P_t(\lambda)}, \mathcal{F}_\mu \chi_{r_2} \overline{P_t(z)} \right)_{L^2(\mu)} &= 2\pi (k_{r_1,\lambda}, k_{r_2,z})_{L^2(\mu)} = 2\pi k_{r,\lambda}(z), \\ &= \left( \chi_{r_1} \overline{P_t(\lambda)}, \chi_{r_2} \overline{P_t(z)} \right)_{L^2[0,2a]}, \end{aligned}$$

where  $r = \min(r_1, r_2)$ . This shows that  $\mathcal{F}_\mu$  is an isometry on  $L$ . Since the linear span of the set  $\{k_{2a,\lambda}, \lambda \in \mathbb{C}\}$  is complete in  $(\text{PW}_{[0,2a]}, \mu)$ , the operator  $\mathcal{F}_\mu$  is unitary. It is also clear from the definition that  $\mathcal{F}_\mu$  maps  $L^2[0, r]$  onto  $(\text{PW}_{[0,r]}, \mu)$  for every  $r \in [0, 2a]$ .  $\square$

**Proof of Theorem 3.** At first, consider a positive bounded invertible operator  $W_\psi$  with real symbol  $\psi \in \mathcal{S}'$  on a finite interval  $[0, a]$ . Let  $\mathcal{F}$  denote the unitary Fourier transform on  $L^2(\mathbb{R})$ . Take a smooth function  $h$  with support in  $(0, a)$  and put  $\hat{f} = \mathcal{F}f$ . Consider the operator  $\hat{W}_\psi = \mathcal{F}W_\psi\mathcal{F}^{-1}$  on  $\text{PW}_{[0,a]}$ . We have  $(\hat{W}_\psi \hat{h}, \hat{h})_{L^2(\mathbb{R})} = \langle \hat{\psi}, |\hat{h}|^2 \rangle_{\mathcal{S}'}$ , where  $\hat{\psi}$  is the Fourier transform of the tempered distribution  $\psi$ . It follows that

$$(\hat{W}_\psi f, f)_{L^2(\mathbb{R})} = (\hat{W}_\psi \frac{z-i}{z+i} f, \frac{z-i}{z+i} f)_{L^2(\mathbb{R})}$$

on a dense subset of the set  $Z_{-i} = \{f \in \text{PW}_{[0,a]} : f(-i) = 0\}$ . Since  $\hat{W}_\psi$  is bounded on  $\text{PW}_{[0,a]}$ , we have the last identity for all  $f \in Z_{-i}$ . Hence, the operator  $\hat{W}_\psi$  satisfies assumptions of Lemma 3.1 and we can find a positive Borel measure  $\mu$  on  $\mathbb{R}$  such that

$$(\hat{W}_\psi f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f \bar{g} d\mu$$

for all  $f, g \in \text{PW}_{[0,a]}$ . As in the proof of Theorem 1, we can assume that the measure  $\mu$  is even. Indeed, since  $\psi$  is real, we have  $(\hat{W}_\psi f, f) = (\hat{W}_\psi f^*, f^*)$  for arbitrary  $f \in \text{PW}_{[0,a]}$  and its reflection  $f^* : x \mapsto f(-x)$ . By the assumption, the operator  $\hat{W}_\psi$  is positive, bounded and invertible on  $\text{PW}_{[0,a]}$ . Hence the measure  $\mu$  satisfies (1) for some  $c_1, c_2$  and  $a/2$  in place of  $a$ . By Theorem 2, there is a unitary operator  $\mathcal{F}_\mu : L^2[0, a] \rightarrow (\text{PW}_{[0,a]}, \mu)$  such that  $\mathcal{F}_\mu : L^2[0, r] \rightarrow (\text{PW}_{[0,r]}, \mu)$  for every  $r \in [0, a]$ . Identifying Hilbert spaces  $(\text{PW}_{[0,a]}, \mu)$  and  $\text{PW}_{[0,a]}$  as sets, we can define the operator  $A = \mathcal{F}_\mu^{-1} \mathcal{F}$  on  $L^2[0, a]$ . By construction, the operator  $A$  is bounded and invertible and  $AL^2[0, r] = L^2[0, r]$  for every  $r \in [0, a]$ . We also have

$$(W_\psi h, h)_{L^2[0,a]} = \int_{\mathbb{R}} |\hat{h}|^2 d\mu = (\mathcal{F}h, \mathcal{F}h)_{L^2(\mu)} = (\mathcal{F}_\mu^{-1} \mathcal{F}h, \mathcal{F}_\mu^{-1} \mathcal{F}h)_{L^2[0,a]} \quad (28)$$

for all smooth functions  $h$  with support in  $(0, a)$ . It follows that the operator  $W_\psi$  admits the triangular factorization  $W_\psi = A^* A$ .

It remains to consider the case where  $W_\psi$  is a positive bounded invertible Wiener-Hopf operator on  $L^2[0, \infty)$  with real symbol  $\psi \in \mathcal{S}'$ . It is known (see Section 4.2.7 in [14]) that in this case the Fourier transform of the distribution  $\psi$  coincides with a function  $\sigma$  on  $\mathbb{R}$  such that  $c_1 \leq \sigma(x) \leq c_2$  for some positive constants  $c_1, c_2$  and almost all  $x \in \mathbb{R}$ . In particular, the measure  $\mu = \sigma dm$  is sampling for all Paley-Wiener spaces  $\text{PW}_{[0,r]}$ ,  $r > 0$ . Since  $\psi$  is real, the function  $\sigma$  is even. For every  $r > 1$  we can use Theorem 1 and find a Hamiltonian  $\mathcal{H}_r$  on  $[0, r]$  such that  $\det \mathcal{H}_r(t) = 1$  for almost all  $t \in [0, r]$  and  $\mu$  is the spectral measure for  $\mathcal{H}_r$ . Since the Hamiltonian  $\mathcal{H}$  in Theorem 1 is defined uniquely, we have  $\mathcal{H}_r(t) = \mathcal{H}_{r'}(t)$  for

almost all  $t \in [0, \min(r, r')]$ . This shows that there is the Hamiltonian  $\mathcal{H}$  on  $[0, \infty)$  such that  $\det \mathcal{H} = 1$  almost everywhere and  $\mu$  is the spectral measure for  $\mathcal{H}$ . In particular, we can define a family of entire functions  $\{P_t\}_{t \geq 0}$  such that the mapping

$$\mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^r f(t) P_t(z) dt \quad (29)$$

sends unitarily the space  $L^2[0, r]$  onto the space  $(\text{PW}_{[0, r]}, \mu)$  for every  $r > 0$ , see the proof of Theorem 2. Let  $H_\mu^2(\mathbb{C}_+)$  be the weighted Hardy space with the inner product  $(f, g)_{H_\mu^2(\mathbb{C}_+)} = (f, g)_{L^2(\mu)}$ . Since  $c_1 \leq \sigma \leq c_2$  on  $\mathbb{R}$ , the space  $H_\mu^2(\mathbb{C}_+)$  coincides as a set with the standard Hardy space  $H^2(\mathbb{C}_+) = \mathcal{FL}^2[0, \infty)$ . Define the unitary operator  $\mathcal{F}_\mu$  from  $L^2[0, \infty)$  to  $H_\mu^2(\mathbb{C}_+)$  by formula (29) with  $r = \infty$  on the dense set of compactly supported bounded functions in  $L^2[0, \infty)$ . Then the operator  $A = \mathcal{F}_\mu^{-1} \mathcal{F}$  on  $L^2[0, \infty)$  is bounded and invertible. Moreover,  $AL^2[0, r] = L^2[0, r]$  for every  $r \geq 0$ , and  $W_\psi = A^*A$ , see formula (28).  $\square$

**Remark.** It can be shown that positive bounded invertible Wiener-Hopf operators  $W_\psi$  on  $L^2[0, a]$  with real symbols  $\psi \in \mathcal{S}'$  admit triangular factorisation in the reverse order,  $W_\psi = AA^*$ . In the case  $a = \infty$  the classical Wiener-Hopf factorization works: one can take  $A = \mathcal{F}^{-1} T_{\varphi_\sigma} \mathcal{F}$ , where  $T_{\varphi_\sigma}$  is the Toeplitz operator on  $H^2(\mathbb{C}_+)$  with analytic symbol  $\varphi_\sigma$  such that  $|\varphi_\sigma|^2 = \sigma = \mathcal{F}\psi$ . If  $a > 0$  is finite, then we can use Theorem 3 to find left triangular factorization  $W_\psi = \tilde{A}^* \tilde{A}$  and then put  $A = C_a \tilde{A} C_a$ , where  $C_a : f \mapsto \overline{f(a-x)}$  is the conjugate-linear isometry on  $L^2[0, a]$ . Since  $C_a W_\psi C_a = W_\psi$  for the self-adjoint Wiener-Hopf operator  $W_\psi$  on  $L^2[0, a]$ , and  $C_a^2 = I$ , we have  $W_\psi = AA^*$ . It is also clear that the operator  $A$  is upper-triangular.

## 5. APPENDIX. TWO RESULTS BY L. A. SAKHNOVICH

In paper [17] L. A. Sakhnovich proved (see Theorem 4.1 and Remark 4.1 in [17]) that positive bounded invertible Wiener-Hopf operator

$$T : f \mapsto f - \mu \int_0^\infty f(t) \frac{\sin \pi(t-x)}{\pi(t-x)} dt, \quad f \in L^2[0, \infty), \quad 0 < \mu < 1, \quad (30)$$

densely defined on  $L^2[0, \infty)$  does not admit triangular factorization  $T = A^*A$ , where a bounded invertible operator  $A$  on  $L^2[0, \infty)$  is such that  $AL^2[0, r] = L^2[0, r]$  for every  $r \geq 0$ . Clearly, this assertion contradicts Theorem 3. Let us point out an error in its proof.

The argument in [17] crucially uses the following claim. Let  $\chi_{[-\pi, \pi]}$  be the indicator function of the interval  $[-\pi, \pi]$ . Formulas (4.1) – (4.4) in [17] for  $n = 0$  and  $a_0 = \pi$  determine the function  $\sigma' : x \mapsto \frac{1}{2\pi}(1 - \mu \cdot \chi_{[-\pi, \pi]}(x))$  on  $\mathbb{R}$ . The function

$$\Pi(z) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{1}{2i\pi} \int_{-\infty}^\infty \frac{1+tz}{(z-t)(1+t^2)} \log \sigma'(t) dt \right)$$

from formula (4.10) of [17] (see also formula (4.12) therein) is claimed to satisfy the following relation (formula (4.18) in [17]):

$$\lim_{y \rightarrow +0} \Pi(iy) = \sqrt{1 - \mu}.$$

However, this fact is false. Indeed, we have

$$\frac{1}{\pi i} \frac{1+tz}{(z-t)(1+t^2)} = -\frac{1}{\pi i} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right)$$



and hence  $\sqrt{2\pi}\Pi(z)$  is the outer function in  $\mathbb{C}_+$  whose absolute value on  $\mathbb{R}$  coincides with  $(\sigma')^{-1/2}$  almost everywhere on  $\mathbb{R}$ . Since  $(\sigma')^{-1/2}$  is regular (in fact, constant) near the origin, we have

$$\lim_{y \rightarrow +0} \Pi(iy) = \frac{1}{\sqrt{2\pi}} (\sigma')^{-1/2}(0) = \frac{1}{\sqrt{1-\mu}}.$$

We also would like to note that the last relation agrees well with the first identity in formula (4.19) from [17].

The second part of this section concerns factorization problem for truncated Toeplitz operators generated by general sampling measures for the space  $\text{PW}_a$  not necessarily symmetric with respect to the origin. The result is equivalent to Theorem 4.2 in [16]. The proof below seems to be a bit more straightforward than the original one, possibly, because we consider the one-dimensional situation.

**Proposition 5.1.** *Let  $\mathcal{H}$  be a Hamiltonian on  $[0, \ell]$  such that  $\int_0^\ell \text{trace } \mathcal{H}(r) < \infty$ , and let  $\mu$  be a spectral measure for problem (2). Set  $a = \int_0^\ell \sqrt{\det \mathcal{H}(r)} dr$ . Assume that  $\mu$  satisfies (1). The following assertions are equivalent:*

- (a)  $\det \mathcal{H} > 0$  almost everywhere on  $[0, \ell]$ ;
- (b) there exists a unitary operator  $V_\mu : \text{PW}_a \rightarrow (\text{PW}_a, \mu)$  such that for every  $r \in [0, a]$  we have  $V_\mu \text{PW}_r = (\text{PW}_a, \mu)$ .
- (c) there exists a bounded invertible operator  $A$  on  $\text{PW}_a$  such that  $T_{\mu,a} = A^* A$  and for every  $r \in [0, a]$  we have  $A \text{PW}_r = \text{PW}_r$ .

Given a Hamiltonian  $\mathcal{H}$  on  $[0, \ell]$  such that  $a = \int_0^\ell \sqrt{\det \mathcal{H}(t)} dt > 0$ , we define continuous from the left function  $\xi_\mathcal{H}$  from  $[0, a]$  to  $[0, \ell]$  by

$$r = \int_0^{\xi_\mathcal{H}(r)} \sqrt{\det \mathcal{H}(t)} dt, \quad r \in [0, a].$$

This function is continuous if and only if there are no interval  $(r_1, r_2) \subset [0, \ell]$  such that  $\det \mathcal{H}(t) = 0$  for almost all  $t \in (r_1, r_2)$ . The function  $\xi_\mathcal{H}$  is absolutely continuous if and only if  $\det \mathcal{H}(t) > 0$  for almost all  $t \in [0, a]$ , see Exercise 13 in Chapter IX of [13].

**Proof of Proposition 5.1.** (a)  $\Rightarrow$  (b). Since  $\det \mathcal{H} > 0$  almost everywhere on the interval  $[0, \ell]$ , the function  $\xi = \xi_\mathcal{H}$  is absolutely continuous and

$$\xi'(r) = \frac{1}{\sqrt{\det \mathcal{H}(\xi(r))}}$$

for almost all  $r \in [0, a]$ . Consider the Hamiltonian  $\tilde{\mathcal{H}} : r \mapsto \xi'(r) \mathcal{H}(\xi(r))$  on the interval  $[0, a]$ . We have  $\det \tilde{\mathcal{H}} = 1$  and  $\Theta_{\tilde{\mathcal{H}}}(r, z) = \Theta_\mathcal{H}(\xi(r), z)$  on  $[0, a]$ . Changing variable in (3), we see that  $\mathcal{B}(\tilde{\mathcal{H}}, r) = \mathcal{B}(\mathcal{H}, \xi(r))$  for every  $r \in [0, a]$ , hence  $\mu$  is the spectral measure for  $\tilde{\mathcal{H}}$ . Consider the Weyl-Titchmarsh transforms generated by Hamiltonians  $\tilde{\mathcal{H}}$  and  $\mathcal{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , correspondingly,

$$\mathcal{W}_{\tilde{\mathcal{H}},a} : L^2(\tilde{\mathcal{H}}, a) \rightarrow \mathcal{B}(\tilde{\mathcal{H}}, a), \quad \mathcal{W}_{\mathcal{H}_0,a} : L^2(\mathcal{H}_0, a) \rightarrow \text{PW}_a.$$

Define the operator  $V_\mu : \text{PW}_a \rightarrow \mathcal{B}(\tilde{\mathcal{H}}, a)$  by  $V_\mu = \mathcal{W}_{\tilde{\mathcal{H}},a} \mathbb{M}_{\tilde{\mathcal{H}}^{-1/2}} \mathcal{W}_{\mathcal{H}_0,a}^{-1}$ , where  $\mathbb{M}_{\tilde{\mathcal{H}}^{-1/2}} : L^2(\mathcal{H}_0, a) \rightarrow L^2(\tilde{\mathcal{H}}, a)$  is the multiplication operator by  $\tilde{\mathcal{H}}^{-1/2}$ , that is,  $\mathbb{M}_{\tilde{\mathcal{H}}^{-1/2}} : X \mapsto \tilde{\mathcal{H}}^{-1/2} X$ . Since  $\mathbb{M}_{\tilde{\mathcal{H}}^{-1/2}}$  is unitary, the operator  $V_\mu$  is unitary as

well. It is also clear that  $V_\mu \text{PW}_r = \mathcal{B}(\tilde{\mathcal{H}}, r)$  for every  $r \in [0, a]$ . Using Lemma 3.3, we see that  $\mathcal{B}(\tilde{\mathcal{H}}, r) = (\text{PW}_r, \mu)$ , as required.

(b)  $\Rightarrow$  (a). We will show that the function  $\xi = \xi_{\mathcal{H}}$  is absolutely continuous. Let  $\chi_r$  be the indicator function of an interval  $[0, r]$ . For every  $r \in [0, a]$  consider the functions  $X_{\xi(r)} = \chi_{\xi(r)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $Y_{\xi(r)} = \chi_{\xi(r)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  in  $L^2(\mathcal{H}, \xi(r))$ . A straightforward modification of Lemma 3.3 gives  $\mathcal{B}(\mathcal{H}, \xi(r)) = (\text{PW}_r, \mu)$  for all  $r \in [0, a]$ . Put

$$X_r^0 = \mathcal{W}_{\mathcal{H}_0, a}^{-1} V_\mu^{-1} \mathcal{W}_{\mathcal{H}, a} X_{\xi(r)}, \quad Y_r^0 = \mathcal{W}_{\mathcal{H}_0, a}^{-1} V_\mu^{-1} \mathcal{W}_{\mathcal{H}, a} Y_{\xi(r)}.$$

Since  $V_\mu$  is isometric and  $V_\mu \text{PW}_r = (\text{PW}_r, \mu)$ , we have  $\mathcal{P}_{\mu, r} V_\mu = V_\mu \mathcal{P}_r$ , where  $\mathcal{P}_r$ ,  $\mathcal{P}_{\mu, r}$  are the orthogonal projections on  $\text{PW}_a$ ,  $(\text{PW}_a, \mu)$ , with ranges  $\text{PW}_r$ ,  $(\text{PW}_r, \mu)$ , respectively. It follows that  $X_r^0 = \chi_r X_a^0$  and  $Y_r^0 = \chi_r Y_a^0$ . Using the fact that the operators  $\mathcal{W}_{\mathcal{H}_0, a}$ ,  $\mathcal{W}_{\mathcal{H}, a}$  are unitary, we obtain

$$\begin{aligned} \int_0^{\xi(r)} \text{trace } \mathcal{H}(t) dt &= \int_0^{\xi(r)} \left( \langle \mathcal{H}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle + \langle \mathcal{H}(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle \right) dt, \\ &= \|X_{\xi(r)}\|_{L^2(\mathcal{H}, \ell)}^2 + \|Y_{\xi(r)}\|_{L^2(\mathcal{H}, \ell)}^2, \\ &= \|X_r^0\|_{L^2(\mathcal{H}_0, a)}^2 + \|Y_r^0\|_{L^2(\mathcal{H}_0, a)}^2, \\ &= \|\chi_r X_a^0\|_{L^2(\mathcal{H}_0, a)}^2 + \|\chi_r Y_a^0\|_{L^2(\mathcal{H}_0, a)}^2, \\ &= \int_0^r \left( \|X_a^0(t)\|_{\mathbb{C}^2}^2 + \|Y_a^0(t)\|_{\mathbb{C}^2}^2 \right) dt. \end{aligned} \quad (31)$$

The above equalities hold for all  $r \in [0, a]$ . Let us define the function  $\kappa$  on  $[0, \ell]$  by

$$\kappa(s) = \int_0^s \text{trace } \mathcal{H}(t) dt, \quad s \in [0, \ell].$$

Then  $\kappa$  is an absolutely continuous function with positive derivative almost everywhere on  $[0, \ell]$ , hence the inverse mapping  $\kappa^{-1}$  is also absolutely continuous and has positive derivative. On the other hand, formula (31) shows that  $\kappa(\xi)$  is an absolutely continuous function. It follows that the superposition  $\xi = \kappa^{-1}(\kappa(\xi))$  is absolutely continuous and hence  $\det \mathcal{H} > 0$  almost everywhere on  $[0, \ell]$ .

(b)  $\Rightarrow$  (c). Since  $\mu$  satisfies (1), the identical embedding  $j : \text{PW}_a \rightarrow (\text{PW}_a, \mu)$  is a bounded and invertible operator. Define  $A = V_\mu^{-1} j$ . Then for all  $f, g$  in  $\text{PW}_a$  we have

$$(A^* A f, g) = (V_\mu^{-1} j f, V_\mu^{-1} j g)_{L^2(\mathbb{R})} = (j f, j g)_{L^2(\mu)} = \int_{\mathbb{R}} f \bar{g} d\mu = (T_{\mu, a} f, g), \quad (32)$$

by the unitarity of the operator  $V_\mu$ . It follows that  $T_{\mu, a} = A^* A$ . By construction, the operator  $A$  is invertible. We also have  $A \text{PW}_r = \text{PW}_r$  for all  $r \in [0, a]$ , hence  $A$  is upper-triangular.

(c)  $\Rightarrow$  (b). Assume that  $T_{\mu, a}$  admits a left triangular factorization  $T_{\mu, a} = A^* A$ . Define the operator  $V_\mu : \text{PW}_a \rightarrow (\text{PW}_\mu, a)$  by  $V_\mu = j A^{-1}$ , where  $j$  is the embedding from  $\text{PW}_a$  to  $(\text{PW}_a, \mu)$ . Then  $V_\mu \text{PW}_r = (\text{PW}_r, \mu)$  for every  $r \in [0, a]$  and

$$\begin{aligned} (V_\mu f, V_\mu g)_{L^2(\mu)} &= ((A^{-1})^* j^* j A^{-1} f, g)_{L^2(\mathbb{R})} \\ &= ((A^*)^{-1} T_{\mu, a} A^{-1} f, g)_{L^2(\mathbb{R})} = (f, g)_{L^2(\mathbb{R})}, \end{aligned}$$

where we used the identity  $T_{\mu, a} = j^* j$ , see (32). Since  $A$  and  $j$  are invertible,  $V_\mu$  is a unitary operator.  $\square$

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